Chapter 2

POLYNOMIAL EQUATIONS AND INEQUALITIES

It's happened to everyone. You've lost your favourite CD, and your room is an unbelievable mess. Rather than attempting to sort through everything, why not consider a few key places where it could be, and examine these areas closely until you find your CD. Similarly, if a manufacturer discovers a flaw in her product, the key intermediate assembly stages are examined individually until the source of the problem is found. These are two examples of a general learning and problem solving strategy: consider a thing in terms of its component parts, without losing sight of the fact that the parts go together. This problem solving strategy is a great way to solve mathematical equations, as well. In this chapter, you will see that polynomial equations can be solved using the same strategy you might use for finding a lost CD. Just examine the key component factors until you solve the problem!

CHAPTER EXPECTATIONS In this chapter, you will

- understand the Remainder and Factor Theorems, **Section 2.1**
- factor polynomial expressions, **Section 2.2**
- compare the nature of change in polynomial functions with that of linear and quadratic functions, **Section 2.3**
- determine the roots of polynomial equations, **Section 2.3**
- determine the real roots of non-factorable polynomial equations, **Section 2.4**
- solve problems involving the abstract extensions of algorithms, **Section 2.4**
- solve factorable and non-factorable polynomial inequalities, **Section 2.5**
- write the equation of a family of polynomial functions, **Section 2.5**
- write the equation of a family of polynomial functions, **Career Link**
- describe intervals and distances, **Section 2.6**

Review of Prerequisite Skills

To begin your study of Polynomial Equations and Inequalities in Chapter 2, you should be familiar with the following skills:

Solving Linear Equations and Inequalities

- $$\begin{aligned} 4(2x - 3) - 2x &= 9 - x \\ 8x - 12 - 2x &= 9 - x \\ 7x &= 21 \\ x &= 3 \end{aligned}$$
- $$\begin{aligned} 2x - 3 &\leq 6x + 13 \\ 2x - 6x &\leq 3 + 13 \\ -4x &\leq 16 \\ 4x &\geq -16 \\ x &\geq -4. \end{aligned}$$

Evaluating Polynomial Functions

- If $f(x) = 2x^3 - 3x + 7$, then $f(2) = 2(2)^3 - 3(2) + 7$
  $= 17.$

Factoring Quadratic Polynomials

- $x^2 - 7x + 12 = (x - 3)(x - 4)$
- $6x^2 - 17x - 14 = (2x - 7)(3x + 2)$

Solving Quadratic Equations by Factoring

- $3x^2 - 5x - 2 = 0$
  $(3x + 1)(x - 2) = 0$
  $3x + 1 = 0$ or $x - 2 = 0$
  $x = -\frac{1}{3}$ or $x = 2$
- $12x^2 + 7x - 10 = 0$
  $(3x - 2)(4x + 5) = 0$
  $3x - 2 = 0$ or $4x + 5 = 0$
  $x = \frac{2}{3}$ or $x = -\frac{5}{4}$

Solving Quadratic Equations Using the Quadratic Formula

- For the equation $ax^2 + bx + c = 0$, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

For $6x^2 - x - 2 = 0$, $a = 6$, $b = -1$, $c = -2$.

$$\begin{aligned} x &= \frac{1 \pm \sqrt{1 - 4(6)(-2)}}{12} \\ &= \frac{1 \pm 7}{12} \\ x &= \frac{2}{3} \text{ or } x = -\frac{1}{2} \end{aligned}$$

For $6x^2 + 2x - 3 = 0$, $a = 6$, $b = 2$, $c = -3$.

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 4(6)(-3)}}{12} \\ &= \frac{-2 \pm \sqrt{76}}{12} \\ &= \frac{-2 \pm 2\sqrt{19}}{12} \\ &= \frac{-1 \pm \sqrt{19}}{6} \\ x &= 0.6 \text{ or } 0.9 \end{aligned}$$

(correct to one decimal)

Exercise

1. Solve.

 a. $3x + 1 = x - 5$
 b. $3(x - 2) + 7 = 3(x - 7)$
 c. $7x - 2(x - 3) = 9x - 5$
 d. $(x + 3)(x - 2) = x^2 - 5x$

2. Solve and graph on the real number line.

 a. $3x - 2 > 2x + 5$
 b. $5x + 4 \geq 7x - 8$
 c. $4x - 5 \leq 2(x - 7)$
 d. $4x + 7 < 9x + 17$

3. Evaluate each of the following for $f(x) = 2x^2 - 3x + 1$.

 a. $f(1)$
 b. $f(-2)$
 c. $f(3)$
 d. $f\left(\frac{1}{2}\right)$

4. Evaluate each of the following for $f(x) = x^3 - 2x^2 + 4x + 5$.

 a. $f(-1)$
 b. $f(2)$
 c. $f(-3)$
 d. $f\left(\frac{1}{2}\right)$

5. Factor each of the following fully.

 a. $x^2 - 14x + 48$
 b. $y^2 - 3y + 2$
 c. $3x^2 - 10x + 7$
 d. $3x^3 - 75x$
 e. $6x^2 + 7x - 3$
 f. $x^3 + x^2 - 56x$
 g. $4x^2 + 20x$
 h. $3x^3 - 12x$
 i. $6x^2 - 14x - 12$

6. Solve.

 a. $x(x + 4) = 0$
 b. $(x - 3)(x + 2) = 0$
 c. $x^2 + 5x + 6 = 0$
 d. $y^2 + 9y + 18 = 0$
 e. $x^2 - 2x - 15 = 0$
 f. $7x^2 + 3x - 4 = 0$
 g. $3x^2 - 10x + 7 = 0$
 h. $x^3 - 9x = 0$
 i. $3x^2 - 13x + 4 = 0$

7. Recall that the quadratic formula to solve the quadratic equation $ax^2 + bx + c = 0, x \in C$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
 Solve correct to one decimal place.

 a. $x^2 + 4x - 8 = 0$
 b. $3y^2 - 5y - 4 = 0$
 c. $3x^2 + x + 3 = 0$
 d. $x^2 - 5x - 4 = 0$
 e. $2x^2 = 3 + 5x$
 f. $6y^2 - 5y = 6$
 g. $2p^2 - 3p + 5 = 0$
 h. $x^2 + 5x - 6 = 0$
 i. $2x(x - 5) = (x + 2)(x - 3)$

Career Link investigate

CHAPTER 2: RESEARCHING DOSE–RESPONSE RELATIONSHIPS

In response to the health concerns of Canada's aging population, the pharmaceutical industry has dramatically increased its investment in research over the past ten years. A key component of the research process is the generation of mathematical models that predict dose-response relationships. "Dose" refers to the quantity of medication administered to a patient, and response refers to the effect on the patient. For example, the dose-response relationship for asthma medication may be in terms of the mass of drug administered versus the percentage increase in lung capacity. Polynomial equations are often used to model the dose-response relationship because they can be fit to a data set that changes slope a number of times and may cross the x-axis multiple times (i.e., it may feature multiple roots). In this chapter, you will develop the algebraic tools to solve polynomial equations and inequalities, then you will investigate the properties of polynomial roots and the absolute-value function.

Case Study — Pharmaceutical Researcher

Pharmaceutical companies are, of course, also interested in modelling the side-effect responses of medication. For example, the equation

$$R(t) = 5\left(t^4 - 4t^3 + \frac{19}{4}t^2 - \frac{3}{2}t\right)$$

can be used to model the side-effect response $[R(t)]$ in degrees Celsius above or below the normal body temperature (36.9°C) of an experimental drug t hours after it was administered. The equation is valid for $0 < t < 2.2$ hours. Due to the stress of temperature change on the body, a second drug is administered at the moment the patient's temperature starts to exceed 36.9°C.

DISCUSSION QUESTIONS

1. Within the context of the problem, what happens when $R(t)$ crosses the t-axis?
2. Using your prior knowledge of linear and quadratic functions and your work in Chapter 1, predict how many times the second drug will have to be administered. (*Hint:* Think about the degree of the function.) Explain using a rough sketch. Do not make a table of values or plot the graph.
3. Once again using your prior knowledge of linear and quadratic functions and your work in Chapter 1, predict how many times the patient's temperature can "spike" (i.e., reach a maximum or minimum). Explain using a rough sketch.

At the end of this chapter you will apply the tools of solving polynomial equations and inequalities in assessing the performance of the experimental drug introduced above.

Section 2.1 — The Factor Theorem

The Remainder Theorem tells us that when we divide $x^2 - 5x + 6$ by $x - 3$, the remainder is

$$\begin{aligned} f(3) &= (3)^2 - 5(3) + 6 \\ &= 9 - 15 + 6 \\ &= 0. \end{aligned}$$

Since the remainder is zero, $x^2 - 5x + 6$ is divisible by $(x - 3)$. By divisible, we mean *evenly* divisible. If $f(x)$ is divisible by $x - p$, we say $x - p$ is a factor of $f(x)$. On the other hand, if we divide $x^2 - 5x + 6$ by $(x - 1)$, the remainder is

$$\begin{aligned} f(1) &= (1)^2 - 5(1) + 6 \\ &= 2. \end{aligned}$$

The fact that the remainder is not zero tells us that $x^2 - 5x + 6$ is not evenly divisible by $(x - 1)$. That is, $(x - 1)$ is not a factor of $x^2 - 5x + 6$.

The Remainder Theorem tells us that if the remainder is zero on division by $(x - p)$, then $f(p) = 0$. If the remainder is zero, then $(x - p)$ divides evenly into $f(x)$, and $(x - p)$ is a factor of $f(x)$. Conversely, if $x - p$ is a factor of $f(x)$, then the remainder $f(p)$ must equal zero. These two statements give us the Factor Theorem, which is an extension of the Remainder Theorem.

The Factor Theorem
$(x - p)$ is a factor of $f(x)$ if and only if $f(p) = 0$.

EXAMPLE 1 Show that $x - 2$ is a factor of $x^3 - 3x^2 + 5x - 6$.

Solution 1

$$\begin{aligned} f(2) &= 2^3 - 3(2)^2 + 5(2) - 6 \\ &= 0 \end{aligned}$$

Since $f(2) = 0$, $x - 2$ is a factor of $x^3 - 3x^2 + 5x - 6$.

Solution 2

Dividing

$$\begin{array}{r@{}l} & x^2 - x + 3 \\ x - 2 & \overline{)\,x^3 - 3x^2 + 5x - 6} \\ & \underline{x^3 - 2x^2} \\ & - x^2 + 5x \\ & \underline{- x^2 + 2x} \\ & 3x - 6 \\ & \underline{3x - 6} \\ & 0 \end{array}$$

Since the remainder is zero, $x - 2$ is a factor of $x^3 - 3x^2 + 5x - 6$. Both solutions verify that $x - 2$ is a factor. Note that Solution 2 tells us that the second factor is $x^2 - x + 3$.

EXAMPLE 2 Is $(x + 2)$ a factor of $f(x) = x^3 + 3x^2 + 5x + 4$?

Solution

We test if $x + 2$ is a factor by evaluating $f(-2)$.

$$\begin{aligned} f(-2) &= (-2)^3 + 3(-2)^2 + 5(-2) + 4 \\ &= -8 + 12 - 10 + 4 \\ &= -2 \end{aligned}$$

Since $f(-2) \neq 0$, $(x + 2)$ is not a factor of $x^3 + 3x^2 + 5x + 4$.

In using the Factor Theorem, we must find a value p so that $f(p) = 0$. We can then say that $(x - p)$ is a factor, and by division we can determine the second factor. The question is how to determine the value of p. This is partly a matter of guessing, but we make the guessing easier by noting that there is a limited number of possible values. In the function $f(x) = x^3 - 4x^2 + 3x - 6$, if there is a value for p such that $f(p) = 0$, then $p^3 - 2p^2 + 3p - 6 = 0$. We are, of course, interested in **integer values** for p. Note that $p^3 - 2p^2 + 3p = 6$ means that $p(p^2 - 2p + 3) = 6$. If p is an integer, then $p^2 - 2p + 3$ is an integer, so the only possible values for p are ± 1, ± 2, ± 3, ± 6, and we need to consider only these. In other words, **the only possible values for p are divisors of the constant term in $f(x)$.**

EXAMPLE 3 Factor $x^3 - x^2 - 14x + 24$.

Solution

Possible values for p are ± 1, ± 2, ± 3, ± 4, ± 6, ± 12, and ± 24.

$$\begin{aligned} f(1) &= 1 - 1 - 14 + 24 \neq 0 \\ f(-1) &= -1 - 1 + 14 + 24 \neq 0 \\ f(2) &= 8 - 4 - 28 + 24 = 0 \end{aligned}$$

Therefore $(x - 2)$ is a factor of $f(x)$.

To find the other factor(s), one method is to use long division and divide $x^3 - x^2 - 14x + 24$ by $x - 2$ as follows:

$$\begin{array}{r} x^2 + x - 12 \\ x - 2\overline{\big)\,x^3 - x^2 - 14x + 24} \\ \underline{x^3 - 2x^2} \\ x^2 - 14x \\ \underline{x^2 - 2x} \\ -12x + 24 \\ \underline{-12x + 24} \\ 0 \end{array}$$

Factoring further, $x^2 + x - 12 = (x + 4)(x - 3)$.
Therefore, $x^3 - x^2 - 14x + 24 = (x - 2)(x + 4)(x - 3)$.

An alternative is to use the following method of comparing coefficients:

$$\begin{aligned} x^3 - x^2 - 14x + 24 &= (x - 2)(x^2 + kx - 12) \\ &= x^3 + kx^2 - 12x - 2x^2 - 2kx + 24 \\ &= x^3 + (k - 2)x^2 + (-12 - 2k)\,x + 24 \end{aligned}$$

Comparing the coefficients of x^2, $k - 2 = -1$, so $k = 1$. A check can be obtained by comparing the coefficients of x, which gives $-12 - 2k = -14$ and $k = 1$. Therefore, $x^3 - x^2 - 14x + 24 = (x - 2)(x^2 + x - 12)$, and further factoring gives $x^3 - x^2 - 14x + 24 = (x - 2)(x - 3)(x + 4)$.

EXAMPLE 4 Factor $f(x) = x^3 + 9x^2 + 5x - 18$.

Solution

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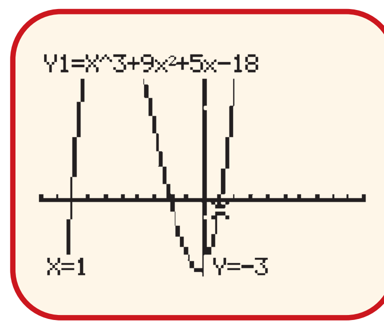
Possible values of p such that $f(p) = 0$ are ± 1, ± 2, ± 3, ± 6, ± 9, and ± 18. Checking all of these is time-consuming. We can help ourselves by using a calculator to sketch the graph $y = f(x)$. From the graph there are three potential integer values: $p = 1$, $p = -2$, and $p = -8$. But $p = -8$ is impossible, since it is not a divisor of 18.



Now
$$\begin{aligned} f(1) &= 1 + 9 + 5 - 18 \\ &= -3 \\ f(-2) &= (-2)^3 + 9(-2)^2 + 5(-2) - 18 \\ &= 0. \end{aligned}$$

Therefore $x + 2$ is a factor of $f(x)$. To find the other factor(s), you can use long division or you can compare coefficients.

$$\begin{array}{r@{}l} & \;x^2 + 7x - 9 \\ x + 2 & \overline{)\,x^3 + 9x^2 + 5x - 18} \\ & \underline{x^3 + 2x^2} \\ & \qquad 7x^2 + 5x \\ & \qquad \underline{7x^2 + 14x} \\ & \qquad\qquad -9x - 18 \\ & \qquad\qquad \underline{-9x - 18} \\ & \qquad\qquad\qquad 0 \end{array}$$

Since $x^2 + 7x - 9$ cannot be factored further,
$x^3 + 9x^2 + 5x - 18 = (x + 2)(x^2 + 7x - 9)$.

EXAMPLE 5 Factor $f(x) = x^4 - 3x^3 - 13x^2 + 3x + 12$.

Solution

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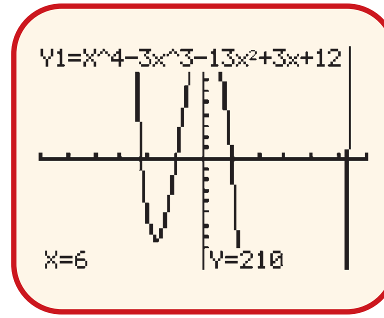
Possible values of p such that $f(p) = 0$ are ± 1, ± 2, ± 3, ± 4, ± 6, and ± 12. From the graph, values to check are $p = 6$, $p = 1$, and $p = -1$. Note that the fourth x-intercept is between -2 and -3 and is not an integer. Graph $y = f(x)$ and use the **1:value** mode under the CALCULATE menu or substitute to evaluate $f(6)$, $f(1)$, and $f(-1)$.



$$\begin{aligned} f(6) &= 6^4 - 3(6)^3 - 13(6)^2 + 3(6) + 12 \\ &= 210 \\ f(1) &= 1 - 3 - 13 + 3 + 12 \\ &= 0 \\ f(-1) &= 1 + 3 - 13 - 3 + 12 \\ &= 0 \end{aligned}$$

Therefore $x - 1$ and $x + 1$ are factors of $f(x)$.
To determine the other factor(s), use the method of comparing coefficients.

$$\begin{aligned} x^4 - 3x^3 - 13x^2 + 3x + 12 &= (x - 1)(x + 1)(x^2 + kx - 12) \\ &= (x^2 - 1)(x^2 + kx - 12) \\ &= x^4 + kx^3 - 12x^2 - x^2 - kx + 12 \\ &= x^4 + kx^3 - 11x^2 - kx + 12 \end{aligned}$$

Since $kx^3 = -3x^3$, $k = -3$.
Now $x^4 - 3x^3 - 13x^2 + 3x + 12 = (x^2 - 1)(x^2 - 3x - 12)$.

All examples considered here involve monic polynomials. A monic polynomial has one as the coefficient of its highest degree term. In the next section, we will consider the use of the Factor Theorem with polynomials having first-term coefficients other than one.

EXAMPLE 6 Factor $x^3 - y^3$.

Solution

Consider this as a function of x. That is, $f(x) = x^3 - y^3$.

$$\begin{aligned} \text{Since } f(y) &= y^3 - y^3 \\ &= 0, \end{aligned}$$

then, by the Factor Theorem, $(x - y)$ is a factor of $x^3 - y^3$. Divide to obtain the other factor(s).

$$\begin{array}{r|l} & x^2 + xy + y^2 \\ \hline x - y & x^3 + 0x^2y + 0xy^2 - y^3 \\ & \underline{x^3 - x^2y} \\ & \quad x^2y + 0xy^2 \\ & \quad \underline{x^2y - xy^2} \\ & \qquad xy^2 - y^3 \\ & \qquad \underline{xy^2 - y^3} \\ & \qquad\qquad 0 \end{array}$$

Therefore $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

The expression $x^3 - y^3$ is referred to as a **difference of cubes** and it occurs often enough that its factorization is worth memorizing:

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

Since $x^3 + y^3 = x^3 - (-y)^3$,

we have $x^3 + y^3 = (x - (-y))(x^2 + x(-y) + (-y)^2$

$= (x + y)(x^2 - xy + y^2)$.

The expression $x^3 + y^3$ is referred to as a **sum of cubes.** By the same process as above, $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

The Sum and Difference of Cubes

$\mathbf{x^3 - y^3 = (x - y)(x^2 + xy + y^2)}$

$\mathbf{x^3 + y^3 = (x + y)(x^2 - xy + y^2)}$

EXAMPLE 7 Factor $27u^3 - 64$.

Solution

Since $27u^3 - 64 = (3u)^3 - (4)^3$

$= (3u - 4)((3u)^2 + (3u)(4) + (4)^2)$

$=(3u - 4)(9u^2 + 12u + 16)$.

Exercise 2.1

Part A

1. If $(x + 8)$ is a factor of $f(x)$, then what is the value of $f(-8)$?

Communication

2. a. If $f(5) = 0$, state a factor of $f(x)$.
   b. Explain how you would find the other factors of $f(x)$.

3. If $f(x) = x^3 + 2x^2 - 5x - 6$ is equal to 0 when $x = -1$ or 2 or -3, what are the factors of $f(x)$? Explain.

4. In each of the following, determine whether the binomial is a factor of $f(x)$.
   a. $x - 1; f(x) = x^2 - 7x + 6$
   b. $x + 2; f(x) = x^2 + 8x + 6$
   c. $x - 2; f(x) = x^3 - 3x^2 - 4x + 12$
   d. $x - 3; f(x) = x^3 + 6x^2 - 2x + 3$
   e. $x + 1; f(x) = x^3 - 5x^2 - 4x + 3$
   f. $2x - 1; f(x) = 4x^3 - 6x^2 + 8x - 3$

Part B

Knowledge/Understanding

5. If $f(x) = x^3 - 2x^2 - 2x - 3$,

a. show that $f(3) = 0$.

b. what is a linear factor of $f(x)$?

c. find the quadratic factor by long division.

6. If $g(x) = x^3 - 2x^2 - 5x + 6$,

a. show that $g(-2) = 0$.

b. what is a linear factor of $g(x)$?

c. find the quadratic factor by the method of comparing coefficients.

Application

7. Completely factor the following:

a. $x^3 - 4x + 3$

b. $x^3 + 2x^2 - x - 2$

c. $y^3 + 19y^2 - 19y - 1$

d. $x^3 + 2x^2 + 5x + 4$

e. $y^3 - y^2 - y - 2$

f. $x^3 - 9x^2 + 22x - 8$

g. $x^4 - 8x^3 + 3x^2 + 40x - 12$

h. $x^4 - 6x^3 - 15x^2 - 6x - 16$

Thinking/Inquiry/Problem Solving

8. If $(x - 1)$ is a factor of $x^3 - 2kx^2 + 3x + 1$, what is the value of k?

9. If $x^3 + 4x^2 + kx - 5$ is divisible by $(x + 2)$, what is the value of k?

Knowledge/Understanding

10. Using the formulas for factoring the sum or difference of cubes, factor each of the following:

a. $x^3 - 27$

b. $y^3 + 8$

c. $125u^3 - 64r^3$

d. $2000w^3 + 2y^3$

e. $(x + y)^3 - u^3z^3$

f. $5u^3 - 40(x + y)^3$

Part C

11. Use the Factor Theorem to prove that $x^3 - 6x^2 + 3x + 10$ is divisible by $x^2 - x - 2$.

Thinking/Inquiry/Problem Solving

12. a. Show that $x - y$ is a factor of $x^4 - y^4$.

b. What is the other factor?

c. Factor $x^4 - 81$.

13. a. Show that $x - y$ is a factor of $x^5 - y^5$.

b. What is the other factor?

c. Factor $x^5 - 32$.

14. a. Show that $x - y$ is a factor of $x^n - y^n$.

    b. What is the other factor?

15. Prove that $(x + a)$ is a factor of $(x + a)^5 + (x + c)^5 + (a - c)^5$.

16. Prove that $(x - a)$ is a factor of $x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc$.

17. If $n \in N$, under what conditions will $(x + y)$ be a factor of $x^n + y^n$?

18. Factor $x^5 + y^5$.

19. Does the expression $x^3 + 2x^2 + 5x + 12$ have any rational factors? Explain.

Section 2.2 — The Factor Theorem Extended

We have seen that if an expression such as $x^3 - 4x^2 + 5x - 6$ has a factor $(x - k)$, where k is an integer, then k must be a divisor of 6. In order to determine which, if any, of the divisors of 6 could be a value for k, we used a graphing calculator to determine a suitable value. In this example, $k = 3$ and $x^3 - 4x^2 + 5x - 6 = (x - 3)(x^2 - x + 2)$. What happens when the coefficient of the highest-order term is an integer other than 1? Let's consider an example.

EXAMPLE 1 Factor $f(x) = 3x^3 - 19x^2 + 27x - 7$.

Solution

If the factors have integers as coefficients, then the first terms must be $3x$, x, and x, and the second terms must be 7, 1, and 1, with the signs in the factors to be determined. We might have a factor such as $(3x - 1)$ or $(3x + 7)$ or $(x - 7)$. Are there other possibilities?

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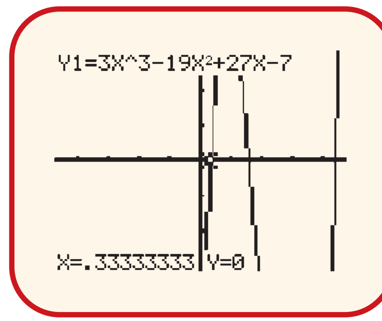

We draw the graph and search for possible values. The graph shown indicates three intercepts, but it is not easy to determine their values. Restricting the domain to $X_{min} = -1$ and $X_{max} = 5$ shows more clearly that there are three possible values for k. They are approximately $k = \frac{1}{3}$, $k = \frac{5}{3}$, and $k = \frac{13}{3}$, since the only denominator we can use is 3 (because 3 and 1 are the only divisors of the coefficient of x^3). Of these, only $\frac{1}{3}$ is a possible value for k, because the numerator must be a divisor of 7. Since $f\left(\frac{1}{3}\right) = 0$, $3x - 1$ is a factor. By long division, or the method of comparing coefficients, $3x^3 - 19x^2 + 27x - 7 = (3x - 1)(x^2 - 6x + 7)$. Since $x^2 - 6x + 7$ has no integer factors, we are done.

A function $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0$ has a factor $(qx - p)$, if $f\left(\frac{p}{q}\right) = 0$, where q divides a_n and p divides a_0.

EXAMPLE 2 Factor $f(x) = 6x^3 - x^2 - 9x - 10$.

Solution

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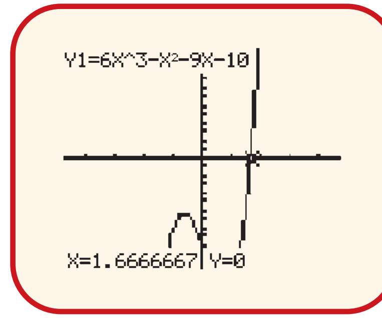
A graph of the function shows that only one value of $k = \frac{p}{q}$ is possible, that it lies between 1 and 2, and that it is close to 2. Since p is a divisor of 10 and q is a divisor of 6, a good guess for k is $\frac{5}{3}$. Evaluating, $f\left(\frac{5}{3}\right) = 0$, so $6x^3 - x^2 - 9x - 10 = (3x - 5)(2x^2 + 3x + 2)$.

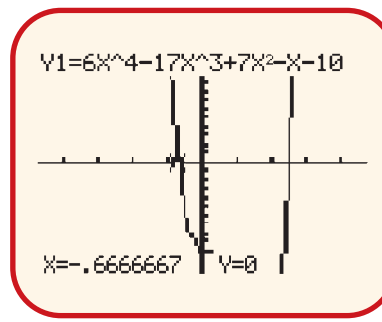


EXAMPLE 3 Factor fully $f(x) = 6x^4 - 17x^3 + 7x^2 - x - 10$.

Solution

technology

A graph of the function shows that $k = \frac{p}{q}$ can be between -1 and 0, or it can be between 2 and 3. Since p divides 10 and q divides 6, we try $k = -\frac{2}{3}$ and $k = \frac{5}{2}$, obtaining $f\left(-\frac{2}{3}\right) = 0$ and $f\left(\frac{5}{2}\right) = 0$. Therefore, two factors are $(3x + 2)$ and $(2x - 5)$. The other factors can be determined by division or by comparison. Then $6x^4 - 17x^3 + 7x^2 - x - 10 = (3x + 2)(2x - 5)(x^2 - x + 1)$, and the third factor cannot be simplified in integers.



Exercise 2.2

Part A

Communication

1. For each of the following, explain how you could find the values $\frac{p}{q}$ of x that potentially could make the polynomial have a value of zero. State all the possible values of $\frac{p}{q}$.

a. $2x^2 + 9x - 5$

b. $3x^3 - 4x^2 + 7x + 8$

c. $4x^3 + 3x^2 - 11x + 2$

d. $8x^3 - 7x^2 + 23x - 4$

e. $6x^3 - 7x^2 + 4x + 3$

f. $2x^3 - 8x^2 + 5x - 6$

Part B

Application

2. A cubic function $f(x)$ with integral coefficients has the following properties: $f\left(\frac{3}{2}\right) = 0$, $(x - 2)$ is a factor of $f(x)$, and $f(4) = 50$. Determine $f(x)$.

3. A cubic function $g(x)$ with integral coefficients has the following properties: $g(3) = 0$, $g\left(-\frac{3}{4}\right) = 0$, $(x + 2)$ is a factor of $g(x)$, $g(1) = -84$. Determine $g(x)$.

Knowledge/ Understanding

4. Factor fully the following polynomials:

a. $2x^3 + x^2 + x - 1$

b. $5x^3 + 3x^2 - 12x + 4$

c. $6x^3 - 17x^2 + 11x - 2$

d. $6x^3 + x^2 - 46x + 15$

e. $5x^4 + x^3 - 22x^2 - 4x + 8$

f. $18x^3 - 15x^2 - x + 2$

g. $3x^4 - 5x^3 - x^2 - 4x + 4$

h. $4x^4 - 19x^3 + 16x^2 - 19x + 12$

Part C

5. Factor fully the following expressions:

a. $px^3 + (p - q)x^2 + (-2p - q)x + 2q$

b. $abx^3 + (a - 2b - ab)x^2 + (2b - a - 2)x + 2$

Section 2.3 — Solving Polynomial Equations

In earlier grades, you learned to solve linear and quadratic equations. A quadratic equation can be solved by factoring (if possible) or by the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. We now consider solutions to higher-order equations. There are formulas for solving a general cubic equation and a general quartic equation, but they are quite complicated. The following examples demonstrate strategies that you can use in solving cubic, quartic, and other higher-order equations. Any equation of the form $f(x) = 0$ can be solved if $f(x)$ can be expressed as a combination of linear and quadratic factors. The first strategy, then, is to factor $f(x)$. It may be possible to factor $f(x)$ by familiar methods, such as grouping terms. If not, we can employ the Factor Theorem.

We will assume that unless otherwise stated, we are to solve all equations using the set of complex numbers, C, as the domain.

In general, if the domain is C, a polynomial equation of degree n has n roots.

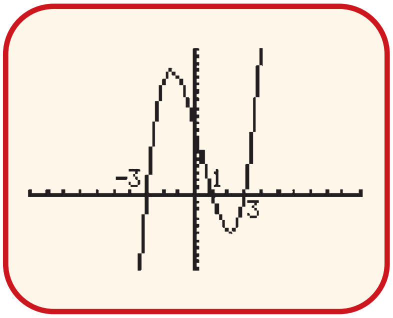
EXAMPLE 1 Solve $x^3 - x^2 - 9x + 9 = 0$.

Solution

The pattern of coefficients $(1, -1, -9, 9)$ suggests grouping the terms.

$$x^3 - x^2 - 9x + 9 = 0$$
$$x^2(x - 1) - 9(x - 1) = 0$$
$$(x - 1)(x^2 - 9) = 0$$
$$(x - 1)(x - 3)(x + 3) = 0$$

Then $x - 1 = 0$ or $x - 3 = 0$ or $x + 3 = 0$
so $x = 1$ or $x = 3$ or $x = -3$.



Note that 1, 3, and -3 are the x-intercepts of the graph of the corresponding cubic function $y = x^3 - x^2 - 9x + 9$.

EXAMPLE 2 Solve $x^3 + 4x - 5 = 0$.

Solution

Since there is no obvious way of grouping, the Factor Theorem is employed.
The factors of 5 are ± 1, and ± 5.
If $f(x) = x^3 + 4x - 5$,
then $f(1) = 1 + 4 - 5 = 0$,
therefore $(x - 1)$ is a factor.

There are two methods we can use for finding the second factor:

Comparing Coefficients

$$\begin{aligned} x^3 + 4x - 5 &= (x-1)(x^2 + kx + 5) \\ &= x^3 + kx^2 + 5x - x^2 - kx - 5 \\ &= x^3 + x^2(k-1) + x(5-k) - 5 \end{aligned}$$

Therefore $k - 1 = 0$ or $5 - k = 4$

$k = 1$ $\qquad$ $k = 1$

The second factor is

$x^2 + x + 5 = 0.$

Using Long Division

$$\begin{array}{r} x^2 + x + 5 \\ x - 1\overline{)x^3 + 0x^2 + 4x - 5} \\ \underline{x^3 - x^2} \\ x^2 + 4x \\ \underline{x^2 - x} \\ 5x - 5 \\ \underline{5x - 5} \\ 0 \end{array}$$

The second factor is

$x^2 + x + 5 = 0.$

Then $x^3 + 4x - 5 = (x-1)(x^2 + x + 5)$

The equation $x^3 + 4x - 5 = 0$

becomes $(x-1)(x^2 + x + 5) = 0$

Then $x - 1 = 0$ or $x^2 + x + 5 = 0$

$$x = 1 \quad \text{or} \quad x = \frac{-1 \pm \sqrt{1 - 20}}{2}$$

$$= \frac{-1 \pm \sqrt{19}i}{2}$$

EXAMPLE 3 Solve $x^3 + 9x^2 + 13x + 5 = 0$.

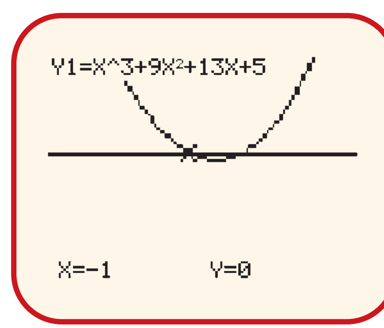
Solution

technology

The graph of $f(x) = x^3 + 9x^2 + 13x + 5$ is shown. If there are integer roots, they must be either ± 1 or ± 5. From the graph, one possible root is -1. Checking, $f(-1) = 0$. Then, by long division or the method of comparison,

$f(x) = (x+1)(x^2 + 8x + 5)$.

Therefore, $x^3 + 9x^2 + 13x + 5 = 0$



and $\qquad (x+1)(x^2 + 8x + 5) = 0$

$$x = -1 \text{ or } x^2 + 8x + 5 = 0$$

$$\begin{aligned} x &= \frac{-8 \pm \sqrt{44}}{2} \\ &= \frac{-8 \pm 2\sqrt{11}}{2} \\ &= -4 \pm \sqrt{11} \end{aligned}$$

The solutions are $x = -1$, $x = -4 \pm \sqrt{11}$.

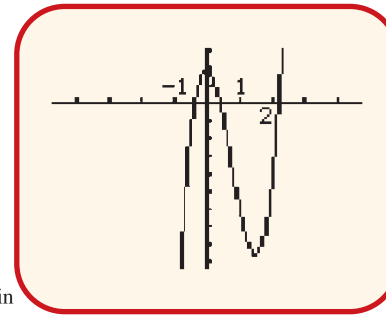
$$\begin{array}{r} x^2 + 8x + 5 \\ x + 1\overline{)x^3 + 9x^2 + 13x + 5} \\ \underline{x^3 + x^2} \\ 8x^2 + 13x \\ \underline{8x^2 + 8x} \\ 5x + 5 \\ \underline{5x + 5} \\ 0 \end{array}$$

EXAMPLE 4 Solve $6x^3 - 13x^2 + x + 2 = 0$.

Solution

technology

Let $f(x) = 6x^3 - 13x^2 + x + 2$.
Since $f(x)$ is not a monic polynomial, non-integer rational roots are possible for $f(x) = 0$. Since p is a divisor of 6 and q is a divisor of 2, the possible values for $\frac{q}{p}$ are ± 1, ± 2, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$, $\pm\frac{1}{6}$, and $\pm\frac{2}{3}$.



Graphing $f(x)$ on a graphing calculator using $X_{\min} = -2$ and $X_{\max} = 2$ as our domain, we find that one root of $f(x) = 0$ lies in the interval $-.34 < x < -.29$, another lies in the interval $.46 < x < .51$, and the third root appears to be 2.

Checking our list of values for $\frac{q}{p}$, we see that the only possible values are $-\frac{1}{3}$, $\frac{1}{2}$, and 2.
Checking, we find that $f\left(-\frac{1}{3}\right) = f\left(\frac{1}{2}\right) = f(2) = 0$.
Therefore $f(x) = (3x + 1)(2x - 1)(x - 2)$ and the roots of $f(x) = 0$ are $-\frac{1}{3}$, $\frac{1}{2}$, and 2.

EXAMPLE 5

a. Find the family of cubic functions whose x-intercepts are -2, 1, and 3.

b. Find the particular member of the above family whose graph passes through the point (2, 20).

Solution

a. Since -2, 1, and 3 are x-intercepts, $(x + 2)$, $(x - 1)$, and $(x - 3)$ must be factors of the cubic function. Therefore, $y = k(x + 2)(x - 1)(x - 3)$, where k is a constant, represents the family of cubic functions.
b. If (2, 20) lies on the graph of one member of the above family, then (2, 20) must satisfy its equation. Substituting, we get $20 = k(4)(1)(-1)$ or $k = -5$. Therefore, the particular cubic function is $y = -5(x + 2)(x - 1)(x - 3)$.

We frequently encounter equations that cannot be factored. In such situations, the best we can hope for is to determine approximate values for the roots using a graphing calculator. The following example illustrates this application.

EXAMPLE 6 Solve $x^3 - 5x^2 + 11 = 0$.

Solution

Can the expression be factored?

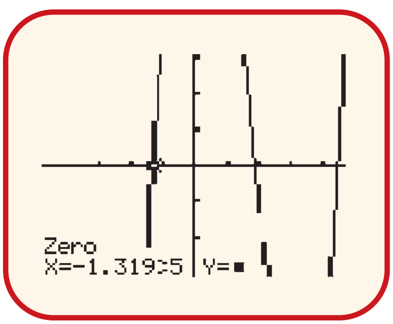
The only factors of 11 are ± 1 and ± 11, and none is a root of the equation.

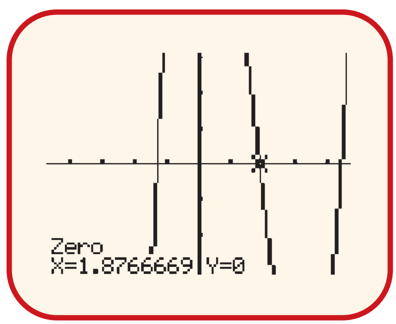
Using a graphing calculator, graph $f(x) = x^3 - 5x^2 + 11$. Using the **ZDecimal** instruction in the ZOOM mode, note from the graph that there is a root slightly to the left of -1.

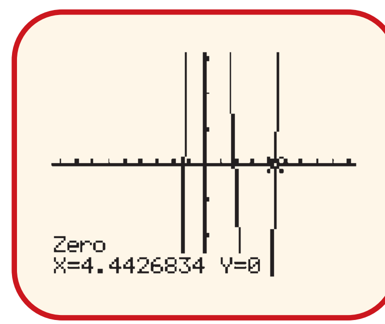
technology

Press [2nd] [CALC TRACE] for the CALCULATE menu.
Select **2:zero** and press [ENTER].
Input -2 as the **Left Bound.**
Input -1 as the **Right Bound.**
Input -1 as the **Guess.**
The approximate root is -1.31935.

Referring back to the graph, you will notice that there is another root close to 2.
Use 1 and 3 as the left and right bounds and 2 as your guess.
The second root is approximately 1.87667.
Use a similar procedure for the other root between 4 and 5.
This third root is approximately 4.44268.







The roots of the equation are -1.3, 1.9, and 4.4, to one decimal place. Note that this procedure will find approximate real roots but will not find the non-real roots (if there are any) of a polynomial equation.

EXAMPLE 7 Solve: $x^4 - 24x^2 - 25 = 0$.

Solution

$(x^2 - 25)(x^2 + 1) = 0$

$x^2 = 25$ or $x^2 = -1$

Then $x = \pm 5$ or $x = \pm i$.

Sometimes making a substitution assists in the solving of higher-order equations. In Example 7, you could let $y = x^2$ to get the equation $y^2 - 24y - 25$. Solve for y and then solve for x. In the next example, the substitution is more subtle.

EXAMPLE 8 Solve $(x^2 - 5x - 5)(x^2 - 5x + 3) = 9$.

Solution

Note the identical $x^2 - 5x$ in the two quadratic factors.
Let $y = x^2 - 5x$, so the equation becomes

$$(y-5)(y+3) = 9$$
$$y^2 - 2y - 24 = 0$$
$$(y - 6)(y + 4) = 0$$
$$y = 6 \quad \text{or} \quad y = -4.$$

Then $x^2 - 5x = 6$ or $x^2 - 5x = -4$

$$x^2 - 5x - 6 = 0 \quad \text{or} \quad x^2 - 5x + 4 = 0$$
$$(x - 6)(x + 1) = 0 \quad \text{or} \quad (x - 4)(x - 1) = 0$$

$x = 6$ or -1 or 4 or 1.

Exercise 2.3

Part A

Communication

1. For the function $f(x) = x^3 + 5x^2 + 2x - 8$, explain how you determine which integral values of x you would use to make $f(x) = 0$.

2. Write a monic polynomial equation with roots 1, -2, and 4.

3. a. Find the family of cubic functions whose x-intercepts are -3, 0, and 2.
   b. Find the particular member of the above family whose graph passes through the point $(-1, 12)$.

4. a. Find the family of cubic functions whose x-intercepts are -2, -1, and 1.
   b. Find the particular member of the above family whose graph passes through the point $(2, -6)$.

5. a. Find the family of quartic functions whose x-intercepts are -2, -1, 1, and 3.
   b. Find the particular member of the above family whose graph passes through the point $(2, -6)$.

6. Write a polynomial equation with integer coefficients that has the roots 1, 2, and $\frac{3}{5}$.

7. If 2 is a root of the equation $2x^3 - 5kx^2 + 7x + 10 = 0$, find the value of k.

Part B

Knowledge/Understanding

8. Solve for x in each of the following, $x \in C$.

a. $x^2 - x - 20 = 0$
b. $x^2 + 2x + 10 = 0$
c. $x(x - 2)(x + 5) = 0$
d. $x(x^2 - 4) = 0$
e. $x^3 = x$
f. $x^4 - 1 = 0$
g. $x^3 - 3x^2 - 4x = 0$
h. $8x^3 - 27 = 0$
i. $x^3 - 3x^2 - 4x + 12 = 0$
j. $x^3 - 9x^2 + 26x = 24$
k. $x^3 - 3x - 2 = 0$
l. $x^3 - 2x^2 - 15x + 36 = 0$
m. $x^3 + 8x + 10 = 7x^2$
n. $x^3 - 3x^2 + 16 = 6x$

9. Solve for x in each of the following, $x \in C$.

a. $2x^3 + 5x^2 - 3x - 4 = 0$
b. $4x^3 + 19x^2 + 11x - 4 = 0$
c. $5x^3 - 8x^2 - 27x + 18 = 0$
d. $4x^4 - 2x^3 - 16x^2 + 8x = 0$
e. $x^4 - 13x^2 + 36 = 0$
f. $x^4 - 7 = 6x^2$
g. $5(x + 1)^3 = -5$
h. $(x + 1)(x + 5)(x + 3) = -3$

10. Solve for x in each of the following, $x \in C$.

a. $x^8 - 10x^4 + 9 = 0$
b. $x^6 - 7x^3 - 8 = 0$
c. $(x^2 - x)^2 - 8(x^2 - x) + 12 = 0$
d. $\left(x - \frac{1}{x}\right)^2 - \frac{77}{12}\left(x - \frac{1}{x}\right) + 10 = 0$
e. $(3x - 5)(3x + 1)^2(3x + 7) + 68 = 0$
f. $(x^2 + 6x + 6)(x^2 + 6x + 8) = 528$

Application

11. A steel cube is uniformly coated with ice. The volume of ice is given by $y = 8x^3 + 36x^2 + 54x$ cm^3, where x is the thickness of ice. Find the thickness of the ice when its volume is 2170 cm^3.

12. Find the approximate roots of the following equations, correct to three decimal places, using a graphing calculator.

a. $x^2 + 7x - 1 = 0$
b. $x^3 - 2x^2 - 8x + 13 = 0$
c. $2x^3 - 6x^2 + 4 = 0$
d. $3x^4 - 20x^2 + 23 = 0$

Application

13. The height, length, and width of a small box are consecutive integers with the height being the smallest of the three dimensions. If the length and width are increased by 1 cm each and the height is doubled, then the volume is increased by 120 cm^3. Find the dimensions of the original small box.

Thinking/Inquiry/ Problem Solving

14. A silo has a cylindrical main section and a hemispherical roof. If the height of the main section is 10 m, what should the radius be in order that the volume of the silo (including the part inside the roof section) is 2000 m^3? (You will need to use your graphing calculator to find the approximate answer correct to two decimal places.)

Part C

15. We start observing a rocket at time $t = 0$, when it has a velocity of 4 km/s (and its displacement is considered to be zero). Its acceleration is 2 km/s^2, and this acceleration is increasing at a rate of 0.6 km/s^2. The displacement of the rocket at time t ($t > 0$) is represented by $s = 0.1t^3 + t^2 + 4t$. At what time has the rocket travelled 25 km?

Section 2.4 — Properties of the Roots of Quadratic Equations

Suppose you are asked to verify that 2 and 7 are the roots of the quadratic equation $x^2 - 9x + 14 = 0$. How would you do it? One way is to substitute each of these values into the left side of the equation and show that the resulting value is zero (the value of the right side). That will certainly work, but is there any other way?

Suppose you are asked to find the quadratic equation whose roots are each five more than the roots of the equation $2x^2 - 17x + 2 = 0$. How could you do that? One way would be to first solve this equation and find its roots. (In this particular case, the roots are not very pretty. They are $\frac{17 \pm \sqrt{273}}{4}$.) Then you would add 5 to each of these numbers (giving you $\frac{37 \pm \sqrt{273}}{4}$, which one still wouldn't describe as pretty) to get the roots of the required new equation. Then you could write the new equation as $\left(x - \frac{37 + \sqrt{273}}{4}\right)\left(x - \frac{37 - \sqrt{273}}{4}\right) = 0$, and finally you could multiply this out and simplify the result, ending with $2x^2 - 37x + 137 = 0$. But here's the good news: there is another way to handle problems such as these, because the roots of a quadratic equation $ax^2 + bx + c = 0$ are related to the coefficients a, b, and c. The investigation below helps us to identify the relationships.

INVESTIGATION For each equation, complete the table, then answer the questions below.

Equation	a	b	c	Roots	Sum of Roots	Product of Roots
$x^2 - 5x + 6 = 0$	1	−5	6			
$x^2 + 3x - 28 = 0$	1	3	−28			
$3x^2 + 19x + 6 = 0$	3	19	6			
$x^2 - 4x + 1 = 0$	1	−4	1			
$2x^2 - 17x + 2 = 0$	2	−17	2			
$5x^2 + x + 2 = 0$	5	1	2			

1. State a relationship between the sum of the roots of a quadratic equation and the coefficients of the equation.
2. State a relationship between the product of the roots of a quadratic equation and the coefficients of the equation.

The results you have noted are easy to prove in general. The quadratic equation $ax^2 + bx + c = 0$ has roots $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-2b + \sqrt{b^2 - 4ac} - \sqrt{b^2 - 4ac}}{2a}$$
$$= -\frac{b}{a}$$

and $$x_1x_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$
$$= \frac{b^2 - (b^2 - 4ac)}{4a^2}$$
$$= \frac{4ac}{4a^2}$$
$$= \frac{c}{a}$$

Also, if x_1 and x_2 are the roots of $ax^2 + bx + c = 0$, then this equation can be written as $(x - x_1)(x - x_2) = 0$, which, when simplified, becomes $x^2 - (x_1 + x_2)x + x_1x_2 = 0$. However, $ax^2 + bx + c = 0$ can also be written as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Then $x^2 - (x_1 + x_2)x + x_1x_2 = x^2 + \frac{b}{a}x + \frac{c}{a}$, and we can conclude that $-(x_1 + x_2) = \frac{b}{a}$ or $(x_1 + x_2) = -\frac{b}{a}$ and $x_1x_2 = \frac{c}{a}$.

The sum of the roots of $ax^2 + bx + c = 0$ is $-\frac{b}{a}$.

The product of the roots of $ax^2 + bx + c = 0$ is $\frac{c}{a}$ and any quadratic equation can be written as
$x^2 - (\text{sum of the roots})x + (\text{product of the roots}) = 0$.

EXAMPLE 1 Find the sum and product of the roots of $3x^2 + 5x + 8 = 0$.

Solution

In this case, $a = 3$, $b = 5$, and $c = 8$.

Therefore, the sum of the roots is $-\frac{b}{a} = -\frac{5}{3}$, and the product of the roots is $\frac{c}{a} = \frac{8}{3}$.

EXAMPLE 2 Find the quadratic equation whose roots are $\frac{1}{2}$ and 2.

Solution

The sum of the roots is $\frac{1}{2} + 2 = \frac{5}{2}$ and the product of the roots is $\left(\frac{1}{2}\right)(2) = 1$.

The quadratic equation is $x^2 - (\text{sum of the roots})\,x + (\text{product of the roots}) = 0$.

Therefore, the equation is $x^2 - \frac{5}{2}x + 1 = 0$ or $2x^2 - 5x + 2 = 0$. It is customary to express the equation with integral coefficients.

EXAMPLE 3 If 4 is one root of the equation $x^2 + kx - 24 = 0$, determine the value of k.

Solution 1

Since 4 is a root, substitute $x = 4$.

$$\begin{aligned} 4^2 + 4k - 24 &= 0 \\ 16 + 4k - 24 &= 0 \\ 4k &= 8 \\ k &= 2 \end{aligned}$$

Solution 2

Let h represent the second root. The product of the roots is $4h$.

Then $4h = \frac{c}{a} = -24$

so $h = -6$,

and the sum of the roots is $-6 + 4 = -\frac{b}{a} = -k$.

Therefore, $k = 2$.

EXAMPLE 4 Find the equation whose roots are each three more than the roots of $x^2 + 7x + 2 = 0$.

Solution

Let x_1 and x_2 represent the roots of the given equation.

Then $x_1 + x_2 = -7$ and $x_1x_2 = 2$.

The roots of the required equation will be $(x_1 + 3)$ and $(x_2 + 3)$.

For the new equation, the sum of the roots is $(x_1 + 3) + (x_2 + 3) = (x_1 + x_2) + 6$

$$\begin{aligned} &= -7 + 6 \\ &= -1 \end{aligned}$$

and the product of the roots is $(x_1 + 3)(x_2 + 3) = x_1x_2 + 3x_1 + 3x_2 + 9$

$$\begin{aligned} &= x_1x_2 + 3(x_1 + x_2) + 9 \\ &= 2 + 3(-7) + 9 \\ &= -10. \end{aligned}$$

Therefore, the required equation is $x^2 - (-1)x + (-10) = 0$ or $x^2 + x - 10 = 0$.

EXAMPLE 5 Find the equation whose roots are the squares of the roots of $3x^2 - 9x + 4 = 0$.

Solution

Let x_1 and x_2 represent the roots of the given equation.

Then $x_1 + x_2 = 3$ and $x_1x_2 = \frac{4}{3}$.

The roots of the required equation are x_1^2 and x_2^2.

The sum of these roots is $(x_1^2 + x_2^2) = (x_1^2 + 2x_1x_2 + x_2^2) - 2x_1x_2$

$$\begin{aligned} &= (x_1 + x_2)^2 - 2x_1x_2 \\ &= (3)^2 - 2\left(\frac{4}{3}\right) \end{aligned}$$

$$= 9 - \frac{8}{3}$$
$$= \frac{19}{3}$$

and the product of these roots is $(x_1^{\,2})(x_2^{\,2}) = (x_1x_2)^2$
$$= \left(\frac{4}{3}\right)^2$$
$$= \frac{16}{9}.$$

The required equation is $x^2 - \frac{19}{3}x + \frac{16}{9} = 0$ or $9x^2 - 57x + 16 = 0$.

Exercise 2.4

Part A

Knowledge/Understanding

1. State the sum and product of the roots of the following equations:

a. $x^2 + 5x + 11 = 0$ b. $2x^2 - 5x + 9 = 0$ c. $3x^2 - 7x - 8 = 0$

Application

2. Find a quadratic equation (with integral coefficients) whose roots have the given sum and product.

a. sum is 3; product is 7
b. sum is -6; product is 4
c. sum is $\frac{1}{5}$; product is $-\frac{2}{25}$
d. sum is $-\frac{13}{12}$; product is $\frac{1}{4}$
e. sum is -11; product is $-\frac{2}{3}$

Knowledge/Understanding

3. Find a quadratic equation (with integral coefficients) having the given roots:

a. 3, 7 b. -5, 8 c. $3, \frac{1}{3}$ d. $\frac{1}{2}, \frac{3}{4}$
e. $-\frac{4}{5}, \frac{3}{25}$ f. $2+i$, $2-i$

Communication

4. If 5 is one root of the equation $2x^2 + kx - 20 = 0$, explain two methods that you would use to find the value of k. Determine k.

Part B

5. If -7 is one root of the equation $x^2 + x - 2k = 0$, determine the other root and the value of k.

6. Find the equation whose roots are each six more than the roots of $x^2 + 8x - 1 = 0$.

7. Find the equation whose roots are each five more than the roots of $2x^2 - 17x + 2 = 0$.

Application

8. Find the equation whose roots are each three times the roots of $3x^2 + 7x + 3 = 0$.

9. Find the equation whose roots are the squares of the roots of $4x^2 - 9x - 2 = 0$.

10. Find the equation whose roots are the reciprocals of the roots of $5x^2 + 10x + 1 = 0$.

11. Find the equation whose roots are the squares of the reciprocals of the roots of $x^2 + 6x - 2 = 0$.

12. Find the equation whose roots are the cubes of the roots of $2x^2 + 4x + 1 = 0$.

Part C

Thinking/Inquiry/Problem Solving

13. A cubic equation may be expressed as $ax^3 + bx^2 + cx + d = 0$ or as $(x - x_1)(x - x_2)(x - x_3) = 0$ where x_1, x_2, and x_3 are the roots of the equation. Use this fact to find the values of $(x_1 + x_2 + x_3)$, $(x_1x_2 + x_1x_3 + x_2x_3)$, and $(x_1x_2x_3)$ in terms of a, b, c, and d.

14. Using the result of Question 13, find a cubic equation (with integral coefficients) whose roots are $\frac{1}{2}$, 2, and 4.

Thinking/Inquiry/Problem Solving

15. Find the equation whose roots are each two more than the roots of $x^3 - 4x^2 + 3x - 2 = 0$.

16. Using the method employed in Question 13, find the relationship between the coefficients in a quartic equation and the roots of that equation.

Section 2.5 — Solving Polynomial Inequalities

When the equal sign in an equation is replaced by any of $>$, $<$, $\geq$, or $\leq$, the equation becomes an inequality. You already know that linear inequalities can be easily solved algebraically. Quadratic and especially cubic or quartic inequalities are more easily solved with the help of a sketch or graph.

EXAMPLE 1

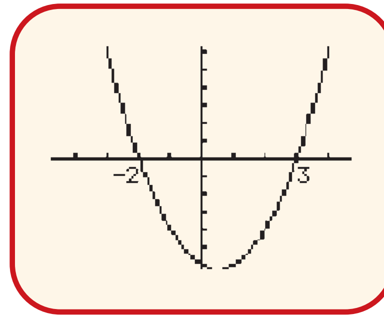
Solve $x^2 - x - 6 < 0$.

technology

Solution

Consider the graph of $y = x^2 - x - 6$
$= (x - 3)(x + 2)$.

The values of x that satisfy the inequality $x^2 - x - 6 < 0$ are the same values for which the graph of $y = x^2 - x - 6$ is below the x-axis. From the graph, $x^2 - x - 6 < 0$ for $-2 < x < 3$.



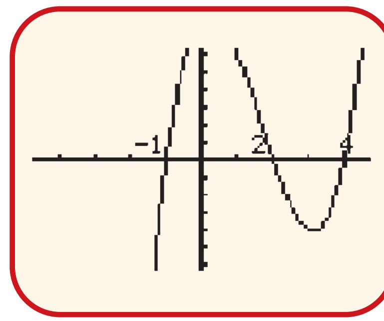
EXAMPLE 2

Solve $x^3 - 5x^2 + 2x + 8 \geq 0$.

technology

Solution

We first graph $y = f(x) = x^3 - 5x^2 + 2x + 8$. From the graph it appears that $x^3 - 5x^2 + 2x + 8 = 0$ if $x = -1$, 2, or 4. Since $f(-1) = f(2) = f(4) = 0$, the x-intercepts are -1, 2, and 4. The solution to the inequality $x^3 - 5x^2 + 2x + 8 \geq 0$ is the set of values of x for which the graph of $y = f(x)$ is on or above the x-axis. The solution is $-1 \leq x \leq 2$ or $x \geq 4$.



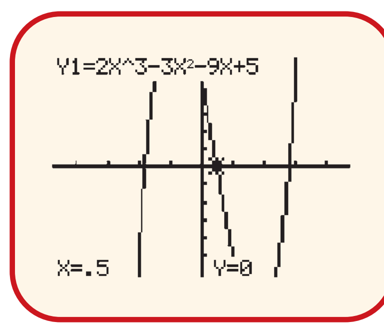
EXAMPLE 3

Solve $2x^3 - 3x^2 - 9x + 5 < 0$.

technology

Solution

The solution to the inequality is the set of values for which the graph of $f(x) = 2x^3 - 3x^2 - 9x + 5$ is below the x-axis. Using the graph and the TRACE function, it appears that the x-intercepts are approximately -1.8, 0.5, and 2.8.



If q is a divisor of 2 and p is a divisor of 5, possible values of x are $\frac{1}{2}$ and $\frac{5}{2}$.

Use **1:value** in the CALCULATE menu or substitution to obtain $f\left(\frac{5}{2}\right) = -5$ and $f\left(\frac{1}{2}\right) = 0$.

Since $f\left(\frac{1}{2}\right) = 0$, $2x - 1$ is a factor.

$$\begin{array}{r} x^2 - x - 5 \\ 2x - 1\overline{)2x^3 - 3x^2 - 9x + 5} \\ \underline{2x^3 - x^2} \\ -2x^2 - 9x \\ \underline{-2x^2 + x} \\ -10x + 5 \\ \underline{-10x + 5} \\ 0 \end{array}$$

Then $2x^3 - 3x^2 - 9x + 5 = 0$

$$(2x - 1)(x^2 - x - 5) = 0$$

$$x = \frac{1}{2} \quad \text{or} \quad x = \frac{1 \pm \sqrt{21}}{2}.$$

Now, from the graph, $2x^3 - 3x^2 - 9x + 5 < 0$ for $x < \frac{1 - \sqrt{21}}{2}$ or $\frac{1}{2} < x < \frac{1 + \sqrt{21}}{2}$.

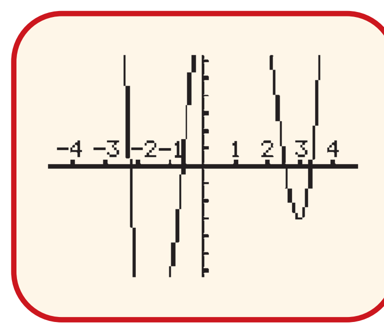
If approximate answers are sufficient, then we can make conclusions without having to solve the equation completely. In the above example, using only the TRACE function, we might, depending on the given conditions, be satisfied with the solution $x < -1.8$, or $0.5 < x < 2.8$. In some situations an approximation is the best we can hope to achieve.

EXAMPLE 4 Solve $x^4 - 3x^3 - 7x^2 + 16x + 12 > 0$.

Solution

technology

The graph of $f(x) = x^4 - 3x^3 - 7x^2 + 16x + 12$ is shown. From the graph, there are no integer roots for the equation $f(x) = 0$. This means that there are no simple factors of the expression and, in fact, it does not factor at all. The best we can do is to approximate the intercepts using the TRACE function. Then, to one decimal place accuracy, $x^4 - 3x^3 - 7x^2 + 16x + 12 > 0$ for $x < -2.2$, or $-0.6 < x < 2.5$, or $x > 3.3$.



Exercise 2.5

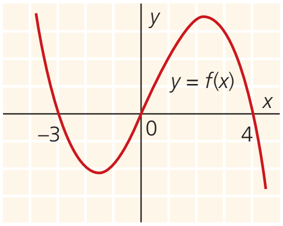
Part A

Knowledge/Understanding

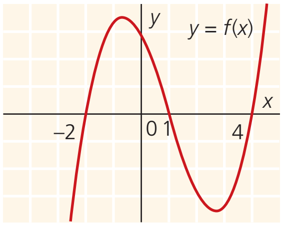
1. Use the graphs of the following functions to state when

(i) $f(x) > 0$ (ii) $f(x) < 0$

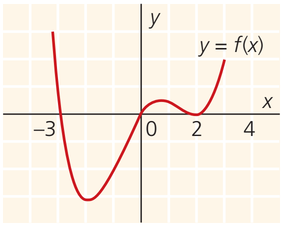
a.



b.



c.



Part B

2. Solve each of the following, $x \in R$.

a. $x(x - 2) < 0$

b. $(x + 3)(x - 1) \leq 0$

c. $x^2 - 7x + 10 \leq 0$

d. $2x^2 + 5x - 3 > 0$

e. $-x^2 + 4x - 4 \geq 0$

f. $-x^3 + 9x \geq 0$

g. $x^3 - 5x^2 < x - 5$

h. $2x^3 + x^2 - 5x + 2 \leq 0$

i. $x^3 - 10x - 2 \geq 0$

j. $x^2 + 1 > 0$

technology

3. The viscosity, v, of oil used in cars is related to its temperature, t, by the formula $v = -t^3 + 9t^2 - 27t + 21$, where each unit of t is equivalent to 50°C.

a. Graph the function of $v = -t^3 + 9t^2 - 27t + 21$ on your graphing calculator.

b. Determine the value of t for $v > 0$, correct to two decimal places.

c. Determine the value of t for $v < -20$, correct to two decimal places.

Application

4. A projectile is shot upwards with an initial velocity of 30 m/s. Its height at time t is given by $h = 30t - 4.9t^2$. During what period of time is the projectile more than 40 m above the ground? Write your answer correct to two decimal places.

Thinking/Inquiry/Problem Solving

5. A rectangular solid is to be constructed with a special kind of wire along all the edges. The length of the base is to be twice the width of the base. The height of the rectangular solid is such that the total amount of wire used (for the whole figure) is 40 cm. Find the range of possible values for the width of the base so that the volume of the figure will lie between 2 cm^3 and 4 cm^3. Write your answer correct to two decimal places.

Section 2.6 — Absolute Value Functions

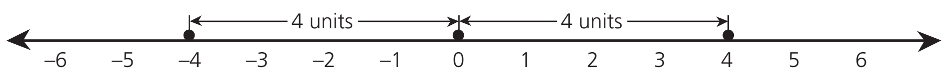
In manufacturing, quality control is very important. If the piston in an engine is too large, it will bind, and if it is too small, it will not work efficiently. The manufacturer might decide that a piston of radius 4 cm must not deviate more than 0.001 cm at this radius.

The margin of error, e, can be written as the inequality $-0.001 \le e \le 0.001$. This inequality can be written by the absolute value statement $|e| \le 0.001$ and read as "the absolute value of e is less than or equal to 0.001."

A real number can be represented by a position on the number line. The absolute value of such a number is the positive distance between the origin and the number. For example, $|4| = 4$, $|-3| = 3$, $|0| = 0$, and so on.

When numbers are represented as points on a number line, $|x|$ is the distance (undirected) from x to 0 (the origin). So $|x| = 4$ means that x is a number four units distant from zero. If $|x| = 4$, x is either 4 or -4.

$$|x| = 4$$



EXAMPLE 1 Evaluate $|5| - |-9| + 3|5 - 12|$.

Solution

$$\begin{aligned} |5| - |-9| + 3|5 - 12| &= 5 - 9 + 3|-7| \\ &= 5 - 9 + 3 \times 7 \\ &= 17 \end{aligned}$$

EXAMPLE 2 Solve for x if $|x - 7| = 3$.

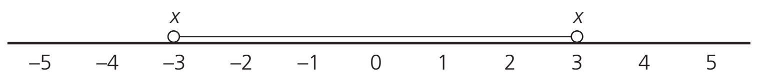
Solution

Since $x - 7$ is three units from the origin, $x - 7 = 3$ or $x - 7 = -3$. Therefore, $x = 10$ or $x = 4$.

EXAMPLE 3 Solve $|x| < 3$.

Solution

Using the geometric definition of absolute value, this statement says that x is less than three units away from the origin. If we look at the real number line, this means that x must lie in the interval between -3 and $+3$.

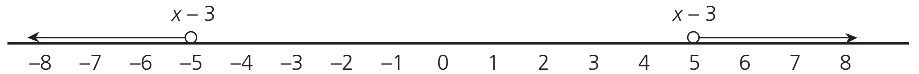


Therefore, the values of x satisfying $|x| < 3$ are $-3 < x < 3$, $x \in R$.

EXAMPLE 4 Solve $|x - 3| > 5$.

Solution

Using the geometric definition of absolute value, this statement says that $x - 3$ is more than five units away from the origin. Looking at the real number line, this means that $x - 3$ must lie in the interval to the right of 5 *or* in the interval to the left of -5.



Therefore, $x - 3 > 5$ or $x - 3 < -5$.
Solving these inequalities for x gives us $x > 8$ or $x < -2$.
We now summarize the preceding discussion by giving the definition of the absolute value of any real number.

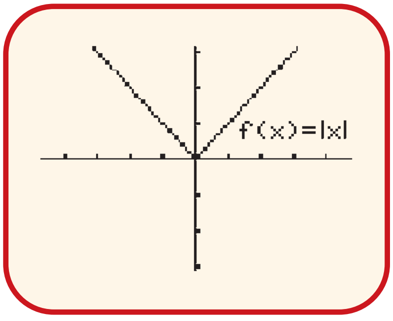
The Absolute Value of x, $x \in R$, is

$$|x| = \begin{cases} x, \text{ if } x \geq 0. \\ -x, \text{ if } x < 0. \end{cases}$$

EXAMPLE 5 Graph the absolute value function $f(x) = |x|$, $x \in R$.

Solution

The graph of $f(x) = |x|$ is shown at the right as it would appear on a graphing calculator or a computer.



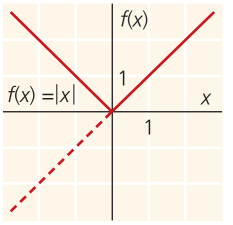
technology

To graph $f(x) = |x|$ on your calculator, first select Y= and then MATH.
In the MATH menu, move the cursor to NUM and select **1:abs(**, the absolute value function.
Select ENTER to return to the y_1 screen.

Enter $y_1 = \text{abs}(x)$ and press GRAPH.

You might also use the following method to graph $f(x) = |x|$.
If $x \geq 0$, $|x| = x$ and the graph is the line $y = x$.
If $x < 0$, $|x| = -x$ and the graph is the line $y = -x$.

Another way of graphing $f(x) = |x|$ is to graph $f(x) = x$, then reflect the portion of the graph below the x-axis in the x-axis, as illustrated, to obtain the required graph.



The definition of absolute value extends to functions, and the algebraic definition can be used in obtaining the graphs of such functions.

For a given function $f(x)$,

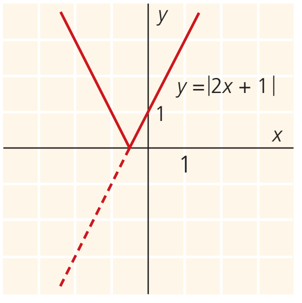
$$|f(x)| = \begin{cases} f(x), \text{ if } f(x) \geq 0. \\ -f(x), \text{ if } f(x) < 0. \end{cases}$$

This means that in order to obtain the graph of $y = |f(x)|$, we can sketch the graph $y = f(x)$ and reflect the portion(s) of the graph that are below the x-axis.

EXAMPLE 6 Graph $y = |2x + 1|$, $x \in R$.

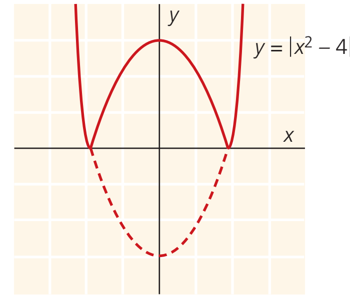
Solution

To graph $y = |2x + 1|$, first graph $y = 2x + 1$. The portion of the graph below the x-axis is reflected in the x-axis as illustrated, to obtain the required graph. The graph $y = |2x + 1|$ is shown.
Note that if $x = 1$, $y = 3$, and if $x = -2$, $y = 3$.
The graph of $y = |2x + 1|$ contains the line $y = 2x + 1$ for $x \geq -\frac{1}{2}$, $x \in R$, and the line $y = -2x - 1$ for $x < -\frac{1}{2}$.



EXAMPLE 7

Graph $y = |x^2 - 4|, x \in R$.



Solution

First graph the parabola $y = x^2 - 4$. Then reflect the portion of the graph that is below the x-axis in the x-axis, as illustrated, to obtain the required graph.

Exercise 2.6

Part A

Knowledge/Understanding

1. Evaluate each of the following:

a. $|-3 - 7|$

b. $|4| + |-15|$

c. $|3| - |-5| + |3 - 9|$

d. $|9 - 3| + 5|-3| - 3|7 - 12|$

2. Graph each of the following on the number line, for $x \in R$. Rewrite each statement without the absolute value bars.

a. $|x| \leq 2$

b. $|x| > 3$

c. $|x| < 4$

d. $|x| \geq 2$

Application

3. Graph each of the following absolute value functions, $x \in R$.

a. $f(x) = |x - 3|$

b. $g(x) = |x + 5|$

c. $h(x) = |2x + 5|$

d. $m(x) = |3x - 6|$

e. $f(x) = |4 - 3x|$

f. $g(x) = |1 - 2x|$

4. Graph each of the following absolute value functions, $x \in R$.

a. $y = |x^2 - 4|$

b. $y = |x^2 - 1|$

c. $y = |x^2 - 2x|$

d. $y = |x^2 + 4x|$

e. $y = |x^3 - 1|$

f. $y = |x^3|$

Part B

Communication

5. In your notebook, describe how you would sketch the graph of the absolute value of a function.

6. Graph each of the following functions:

a. $y = |x^2 - x - 6|$

b. $y = |-2x^2 + 4x - 3|$

c. $y = |x^3 - x|$

7. Solve for x, $x \in R$.

a. $|2x - 1| = 7$ b. $|3x + 2| = 6$ c. $|x - 3| \leq 9$

d. $|x + 4| \geq 5$ e. $|2x - 3| < 4$ f. $|x| = -5$

Part C

technology

8. Use your graphing calculator to solve for x, $x \in R$.

a. $|x| = 3x + 4$ b. $|x - 5| = 4x + 1$

c. $|4x - 8| = 2x$ d. $|x - 1| < x$

e. $|2x + 4| \geq 12x$ f. $|3x - 1| \leq 5|3x - 1| - 16$

g. $|x - 2| + |x| = 6$ h. $|x + 4| - |x - 1| = 3$

Thinking/Inquiry/Problem Solving

9. For which non-zero real numbers, x, is $\frac{|x - |x||}{x}$ a positive integer?

10. Graph $f(x) = \frac{|x - |x||}{x}$ for $x \in R$.

Key Concepts Review

Now that you have completed this chapter, you should be familiar with the following key concepts:

The Factor Theorem

- $(x - p)$ is a factor of $f(x)$ if and only if $f(p) = 0$.

Factor Theorem Extended

- A function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ has a factor $(qx - p)$ if $f\left(\frac{p}{q}\right) = 0$, where q divides a_n and p divides a_0.

Solving Equations

- using grouping, using the Factor Theorem, using the graphing calculator and the Factor Theorem
- finding approximate roots using the graphing calculator
- finding the family of polynomial functions given the x-intercepts of the graph

Properties of the Roots of a Quadratic Equation

- sum of the roots $= -\frac{b}{a}$
- product of the roots $= \frac{c}{a}$

Solving Polynomial Inequalities

- consider when the graph of the polynomial function is above and below the x-axis

Absolute Value

- $|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$
- graphs of $y = |f(x)|$ lie entirely above (or on) the x-axis

Career Link wrap-up

investigate and apply

CHAPTER 2: RESEARCHING DOSE–RESPONSE RELATIONSHIPS

In this task, you will utilize the mathematical model used in the Career Link for predicting the side-effect response to medication.

The model states

$$R(t) = 5\left(t^4 - 4t^3 + \frac{19}{4}t^2 - \frac{3}{2}t\right)$$

where $[R(t)]$ is the temperature in degrees Celsius above or below the normal body temperature of 36.9°C caused by an experimental drug t hours after it was administered. Remember that due to the stress of temperature change on the body, a second drug is administered at the moment the patient's temperature starts to *exceed* 36.9°C. Complete the following questions:

a) Apply the Factor Theorem and the quadratic formula to determine the times when the patient's temperature is at the normal body temperature.

b) Predict when the second drug will be administered by applying your knowledge of solving polynomial inequalities.

c) The company has stated the maximum deviation, above or below normal body temperature, is 2.5°C. Express this statement in algebraic form using absolute-value notation and explaining your work. Describe how you would solve this on the graphing calculator. ●

Review Exercise

1. a. If $f(-3) = 0$, state a factor of $f(x)$.

   b. If $f\left(\frac{2}{3}\right) = 0$, find a factor of $f(x)$, with integral coefficients.

2. a. Find the family of cubic functions whose x-intercepts are 4, 1, and -2.

   b. Find the particular member of the above family whose graph passes through the point (3, 10).

3. a. Determine if $x + 2$ is a factor of $x^5 - 4x^3 + x^2 - 3$.

   b. Determine if $x - 3$ is a factor of $x^3 + x^2 - 11x - 3$.

4. Use the Factor Theorem to factor $x^3 - 6x^2 + 6x - 5$.

5. a. If $x - 1$ is a factor of $x^3 - 3x^2 + 4kx - 1$, what is the value of k?

   b. If $x + 3$ is a factor of $kx^3 + 4x^2 + 2kx - 1$, what is the value of k?

6. Factor each of the following:

   a. $x^3 - 2x^2 + 2x - 1$ b. $x^3 - 6x^2 + 11x - 6$

   c. $8x^3 - 27y^3$ d. $3(x + 2w)^3 - 3p^3r^3$

7. Use the Factor Theorem to prove that $x^2 - 4x + 3$ is a factor of $x^5 - 5x^4 + 7x^3 - 2x^2 - 4x + 3$.

8. Use your graphing calculator to factor each of the following:

   a. $2x^3 + 5x^2 + 5x + 3$ b. $9x^3 + 3x^2 - 17x + 5$

9. If $f(x) = 5x^4 - 2x^3 + 7x^2 - 4x + 8$,

   a. is it possible that $f\left(\frac{5}{4}\right) = 0$? b. is it possible that $f\left(\frac{4}{5}\right) = 0$?

10. Factor fully:

    a. $3x^3 - 4x^2 + 4x - 1$ b. $2x^3 + x^2 - 13x - 5$

    c. $30x^3 - 31x^2 + 10x - 1$

11. Solve for x, $x \in C$.

    a. $x^2 - 3x - 10 = 0$ b. $x^3 - 25x = 0$

    c. $x^3 + 8 = 0$ d. $x^3 - x^2 - 9x + 9 = 0$

    e. $x^4 - 12x^2 - 64 = 0$ f. $x^3 - 4x^2 + 3 = 0$

g. $x^3 - 3x^2 + 3x - 2 = 0$ h. $x^6 - 26x^3 - 27 = 0$

i. $(x^2 + 2x)^2 - (x^2 + 2x) - 12 = 0$

technology

12. Use your graphing calculator to find the approximate roots of the following equations (correct to three decimal places):

a. $x^2 = 2$ b. $x^2 + 10x - 2 = 0$

c. $x^3 - x^2 - 4x - 1 = 0$ d. $2x^3 + x^2 + 2 = 0$

e. $x^4 - 10x^2 + 15 = 0$ f. $x^6 - 11x^5 + x^2 - 1 = 0$

13. If -2 is one root of $x^2 + kx - 6 = 0$, find the other root and the value of k.

14. Find the quadratic equation whose roots are the reciprocals of the roots of $2x^2 + 5x + 1 = 0$.

15. a. State the sum and product of the roots of $2x^2 - x + 4 = 0$.

b. Find a quadratic equation (with integral coefficients) whose roots have a sum of $\frac{1}{15}$ and a product of $-\frac{2}{15}$.

c. Find a quadratic equation (with integral coefficients) whose roots are $3 + 2i$ and $3 - 2i$.

d. If 2 is one root of the equation $3x^2 + 4kx - 4 = 0$, find the other root and the value of k.

e. Find an equation whose roots are each three less than the roots of $x^2 - 5x + 2 = 0$.

f. Find an equation whose roots are the reciprocals of the roots of $2x^2 + x - 4 = 0$.

16. Solve for x, $x \in R$.

a. $(x - 2)(x + 4) < 0$ b. $x^2 + x - 2 \geq 0$ c. $x^3 + 3x \leq 0$

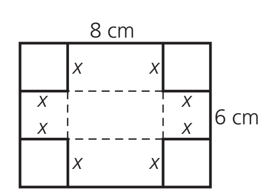
d. $x^3 - 2x^2 - x + 2 > 0$ e. $x^4 \leq 0$ f. $x^4 + 5x^2 + 2 \geq 0$

g. $x^6 - 8x^4 + 2 < 0$ h. $x^9 - 2x^7 + 1 > 0$

17. Solve for x, $x \in R$.

a. $|3x - 1| = 11$ b. $|x + 1| < 3$ c. $|2x - 3| \geq 5$

18. Identical squares are cut from each corner of a rectangular sheet of tin 8 cm $\times$ 6 cm. The sides are bent upward to form an open box. If the volume of the box is 16 cm^3, what is the length of each side of the squares cut from the original sheet?



Chapter 2 Test

Achievement Category	Questions
Knowledge/Understanding	1, 2, 3, 4, 7
Thinking/Inquiry/Problem Solving	8
Communication	6
Application	5, 9

1. Without using long division, determine if $(x + 3)$ is a factor of $x^3 - 5x^2 + 9x - 3$.

2. Factor each of the following:
   a. $x^3 + 3x^2 - 2x - 2$
   b. $2x^3 - 7x^2 + 9$
   c. $x^4 - 2x^3 + 2x - 1$

3. Use your graphing calculator to factor $3x^3 + 4x^2 + 2x - 4$.

4. Solve for x, $x \in C$.
   a. $2x^3 - 54 = 0$
   b. $x^3 - 4x^2 + 6x - 3 = 0$
   c. $2x^3 - 7x^2 + 3x = 0$
   d. $x^4 - 5x^2 + 4 = 0$

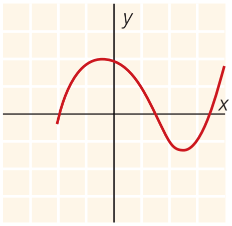
5. Find the quadratic equation whose roots are each three greater than the roots of $x^2 - 2x + 5 = 0$.

6. The Math Wizard states that the x-intercepts of the graph of $f(x) = x^3 + 9x^2 + 26x + 24$ cannot be positive. Is the Math Wizard correct? Explain.

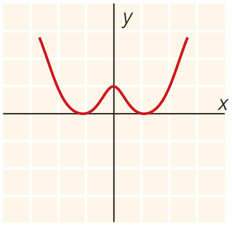
7. Solve for x, $x \in R$.
   a. $(x - 3)(x + 2)^2 < 0$
   b. $x^3 - 4x \geq 0$
   c. $|2x + 5| > 9$

8. What can you deduce about the zeros, the leading coefficient, and the least degree of the polynomial functions represented by the following graphs?

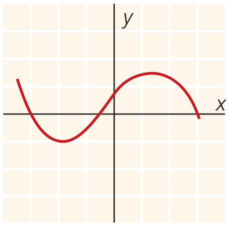
a.



b.



c.



9. The free end of a diving board dips C centimetres when a diver of x kilograms stands on it. The relation is $C = 0.0002x^3 - 0.005x^2 + 0.5x$.

   a. Calculate the amount of dip when a 95 kg diver stands on the board. Give your answer to the nearest tenth of a centimetre.

   b. Calculate the mass of a diver, correct to one decimal place, if the diving board dips 40 cm.